- Declaration: Oct 4,2024
- Project supervisor: Emilia Witkowska, `ewitk@ifpan.edu.pl`
- Consultation: Oct 15, 2024 (Tuesday) at 13:15, hall D
  If needed – another meeting in the week Oct, 14-18
- Presentation: Nov. 22, 2024

This task aims to become familiar with exotic atoms. Most of them are hydrogen-like systems composed of two oppositely charged particles (of exotic or anti-matter nature), electrostatically attracted to each other.

(a) Start by searching the literature on examples of exotic atoms. Discuss the literature.

(b) Prepare a list of known exotic atoms. Compare them, e.g. their lifetimes or conditions of their observations.

Positronium, muonium and hydrogen atom

In what follows we would consider the system called *positronium* which is an atom consisting of an e^+e^- pair. The positron e^+ is the antiparticle of the electron e^-. It is a spin-1/2 particle with the same mass m as the electron but with an electric charge of the opposite sign. The other system considered in this mini-project is *muonium* composed of an electron e^-, and a positive muon, μ^+.

Orbital States

(a) What are the energy levels of the systems, and their degeneracies? What is the Bohr radius a_0 of the two systems? How do they compare with those for hydrogen?

(b) Give the expression for the normalized ground state wave function $\psi_{100}(r)$. Express $|\psi_{100}(r)|^2$ in terms of the fundamental constants: $m, c, \hbar$, and the fine structure constant α.

Hyperfine splitting of the ground state

Consider, as in hydrogen, the spin-spin Hamiltonian in the orbital ground state as:

$$H_{SS} = \frac{A}{\hbar^2}\hat{S}_1\hat{S}_2 \tag{1}$$

where the constant A has the dimension of the energy. As in hydrogen, the constant A originates from the term:

$$A = -\frac{2}{3\epsilon_0 c^2}\gamma_1\gamma_2\hbar^2|\psi_{100}(r)|^2 \tag{2}$$

(a) Recall the eigenstates and eigenvalues of H_{SS} in the spin basis $|\sigma_1, \sigma_2\rangle$.

(b) Explain why the (spin) gyromagnetic ratios of the positron and of the electron have opposite signs: $\gamma_1 = -\gamma_2$. How is it for muon? Express γ's in terms of physical constants.

(c) What is the observed frequency of the hyperfine line of positronium and muonium? How do they compare with the hydrogen one? Justify these differences.

Zeeman Effect in the Ground State

The system is placed in a constant uniform magnetic field B directed along the z axis. The additional Zeeman Hamiltonian has the form

$$H_Z = \omega_1\hat{S}_{1,z} + \omega_2\hat{S}_{2,z} \tag{3}$$

where $\omega_1 = -\gamma_1 B$ and $\omega_2 = -\gamma_2 B$.

(a) Write the action of H_Z on the basis states $|\sigma_1, \sigma_2\rangle$.

(b) Write in terms of A and $\hbar\omega$ the matrix representation of $H = H_{SS} + H_Z$ in the basis of the total spin of the two particles $|S, m\rangle$.

(c) Give the numerical value of $\hbar\omega_{1/2}$ in eV for the field $B = 1T$. Is it easy experimentally to be in a strong field regime, i.e. $\hbar\omega_{1/2} \gg A$?

(d) Calculate the energy eigenvalues in the presence of the field B; express the corresponding eigenstates in the basis $|S, m\rangle$ of the total spin. The largest eigenvalue will be written E^+ and the corresponding eigenstate $|\psi_+\rangle$. For convenience, one can introduce the quantity $x = 8\hbar\omega/(7A)$, and the angle θ defined by $\sin 2\theta = x/\sqrt{1+x^2}, \cos 2\theta = 1/\sqrt{1+x^2}$. Draw qualitatively the variations of the energy levels in terms of B. Are there any remaining degeneracies?

Decay of Positronium

We recall that when a system X is unstable and decays: $X \to Y + \dots$, the probability for this system to decay during the time interval $[t, t+dt]$, if it is prepared at $t = 0$, is $dp = \lambda e^{-\lambda t} dt$, where the decay rate λ is related to the lifetime τ of the system by $\tau = 1/\lambda$. If the decay can proceed via different channels, e.g. $X \to Y + \dots$ and $X \to Z + \dots$, with respective decay rates λ_1 and λ_2, the total decay rate is the sum of the partial rates, and the lifetime of X is $\tau = 1/(\lambda_1 + \lambda_2)$. In what follows, we place ourselves in the rest frame of the positronium.

To determine the hyperfine constant A of positronium, it is of interest to study the energy and the lifetime of the level corresponding to the state $|\psi_+\rangle$ corresponding to the largest eigenvalue as a function of the magnetic field, considered in the previous task. From now on, we assume that the field is weak, i.e. $x = |8\hbar\omega/(7A)| \ll 1$, and we shall make the corresponding approximations.

(a) What are, as a function of x, the probabilities p_S and p_T of finding the state $|\psi_+\rangle$ in the singlet and triplet states, respectively?

(b) Use the result to calculate the decay rates λ_2^+ and λ_3^+ of the state $|\psi_+\rangle$ into two and three photons, respectively, in terms of the parameter x.

(c) What is the lifetime $\tau^+(B)$ of the state $|\psi_+\rangle$? Explain qualitatively its dependence on the applied field B, and calculate $\tau^+(B)$ for $B = 0.4T$.